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Combined Method for Calculating Eigenvector Derivatives with Repeated Eigenvalues

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A combined method for calculating particular solutions of eigenvector derivative governing equations in the generalized eigenproblem with groups of repeated eigenvalues is presented, which finds the first set of the particular solutions by a direct method and the other particular solutions by a family of modal methods. The direct method is based on Gauss elimination, by which the constraint generalized inverse of the frequency shift stiffness matrix can be obtained, as well as the particular solution, and the method is applicable to all nondefective systems. A formula for finding a constraint generalized inverse from another is derived by which the total calculation can be reduced. A simple calculating example is included.

Nomenclature

A, B	= system parameter matrices, defining the generalized eigenproblem
$\bar{A}(\lambda_i)$	= λ_i-frequency shift stiffness matrix, $(A - \lambda_i B)$
$C(\lambda_i)$	= generalized {1} inverse of $\bar{A}(\lambda_i)$
$G_e(\lambda_i)$	= constraint generalized inverse of $\bar{A}(\lambda_i)$
N	= number of degrees of freedom
n, r	= numbers of repeated eigenvalues
p	= design parameter
X	= right eigenvector matrix
$X_\Phi, Y_\Phi, X_\Psi, Y_\Psi$	= matrices of eigenvectors with repeated eigenvalues
x_i	= ith column of X
Y	= left eigenvector matrix
y_i	= ith column of Y
Λ	= eigenvalue matrix
λ_i	= ith eigenvalue
$()\{1\}$	= generalized {1} inverse of a matrix

Superscript

(d)	= dth-order derivative with respect to a design parameter

Introduction

CALCULATION of eigenvector derivatives is widely applied in structural optimal design, system identification, and vibration control. Thus, research of its methods has become more complex. There are many methods for calculating eigenvector derivatives, and they can be categorized into three types: 1) direct method, 2) modal method, and 3) iterative method.[1] This paper involves only the first two types. Nelson[2] presented the direct method first in 1976, and the method was well accepted. But Nelson's method is only applicable to systems without repeated eigenvalues. Mills-Curran[3] made an extension to Nelson's method that is applicable to the case of repeated eigenvalues. But, in practice, Mills-Curran's method still has problems. For example, it must be determined whether the frequency shift stiffness matrix, in which the values of elements may be very small, is nonsingular. That is not easy to do properly with computer calculation. Fox and Kapoor[4] presented the modal method in 1968. But their modal expansion approach requires the complete set of eigenpairs, which is unsuitable for large systems. Although an approximate solution requires a smaller set of eigenpairs, the accuracy may present a problem. Wang[5] provided a modified modal method to improve this approximation. A family of modal methods was advanced by Akgün[6] in 1994. The important feature of Akgün's method is that only eigenpairs that fill up low-frequency bands are needed and only $G_e(0)$, the constraint generalized inverse of stiffness matrix, is required to be computed in the computation of eigenvector derivatives in the low-frequency band. But Akgün's method can only be used with systems with simple eigenvalues. Recently, Wang[7] successfully extended the method to systems with repeated eigenvalues.

This paper derives a formula for computing constraint generalized inverses from an already known one. It is an extension of Wang's[7] formula. The constraint generalized inverse needed in the formula can be a byproduct of solving an eigenvector derivative equation by Gauss elimination. Consequently, a direct method and a modal method can be combined in calculating the eigenvector derivatives of the low-frequency band whereas the eigenvalues may be groups of repeated derivatives. In the case of large systems and where many low-frequency band eigenvector derivatives are wanted, this combined method may yield substantial savings in computation. In addition, this method is applicable to all nondefective systems regardless of symmetry.

Eigenvector Derivatives Equations

The generalized eigenproblem for an N-degree-of-freedom (N-DOF) nondefective system is defined by

$$AX = BX\Lambda, \qquad A^T Y = B^T Y \Lambda \tag{1}$$

where A and B are general complex matrices of order N and B is nonsingular.

The complete biorthogonal right and left eigenvector sets are

$$X = (x_1 \mid x_2 \mid \cdots \mid x_N), \qquad Y = (y_1 \mid y_2 \mid \cdots \mid y_N) \tag{2}$$

They are orthonomalized with respect to B:

$$Y^T B X = I_N \tag{3}$$

The diagonal matrix of the corresponding eigenvalues is

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N), \qquad 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N \tag{4}$$

Suppose

$$\lambda_i = \lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+r-1}, \qquad (\lambda_{i-1} < \lambda_i < \lambda_{i+r}) \tag{5}$$

$$\lambda_j = \lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_{j+n-1}$$
$$(\lambda_{j-1} < \lambda_j < \lambda_{j+n}, \qquad \lambda_i \ne \lambda_j) \tag{6}$$

are two different groups of repeated eigenvalues.

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Let

$$X_\Phi = (x_i \mid x_{i+1} \mid \cdots \mid x_{i+r-1}), \qquad Y_\Phi = (y_i \mid y_{i+1} \mid \cdots \mid y_{i+r-1})$$
$$\Lambda_\Phi = \lambda_i I_r \tag{7}$$

and

$$X_\Psi = (x_j \mid x_{j+1} \mid \cdots \mid x_{j+n-1}), \qquad Y_\Psi = (y_j \mid y_{j+1} \mid \cdots \mid y_{j+n-1})$$
$$\Lambda_\Psi = \lambda_j I_n \tag{8}$$

be, respectively, r-fold and n-fold degenerated eigenpairs to be differentiated. Suppose the eigenvectors are all continuously differentiable (if they were not, they should have been transformed earlier into differentiable[8]).

For the eigenproblem

$$AX_\Phi = BX_\Phi \Lambda_\Phi \tag{9}$$

differentiating it d times with respect to the real parameter p, then letting $p \to p_0$, we have

$$\bar{A}(\lambda_i) X_\Phi^{(d)} = F_d(\lambda_i), \qquad d = 1, 2, \ldots \tag{10}$$

This is the d-order eigenvector derivative governing equation of X_Φ. The coefficient matrix, $\bar{A}(\lambda_i) = A - \lambda_i B$, is called the λ_i-frequency shift stiffness matrix of the original system.

Let

$$\bar{A}_d(\lambda_i) = A^{(d)} - \lambda_i B^{(d)} \tag{11}$$

then, the right-hand side of Eq. (10) can be shown to be

$$F_d(\lambda_i) = \sum_{e=1}^{d} \frac{d!}{e!(d-e)!} \times \left[\sum_{h=1}^{e} \frac{e!}{h!(e-h)!} B^{(e-h)} X_\Phi \Lambda_\Phi^{(h)} - \bar{A}_e(\lambda_i) X_\Phi^{(d-e)} \right] \tag{12}$$

Equation (10) is solvable if and only if

$$Y_\Phi^T F_d = 0, \qquad d = 1, 2, \ldots \tag{13}$$

Substituting Eq. (12) into Eq. (13) gives

$$\Lambda_\Phi^{(d)} = -Y_\Phi^T \Theta_d(\lambda_i), \qquad d = 1, 2, \ldots \tag{14}$$

where

$$\Theta_d(\lambda_i) = F_d(\lambda_i) - BX_\Phi \Lambda_\Phi^{(d)} \tag{15}$$

In particular, when $d = 1$, Eq. (14) becomes

$$\Lambda_\Phi^{(1)} = Y_\Phi^T \bar{A}_1(\lambda_i) X_\Phi \tag{16}$$

The general solution set of Eq. (10) can be written as

$$X_\Phi^{(d)} = \tilde{X}_\Phi^{(d)} + X_\Phi S_d(\lambda_i) \tag{17}$$

where $\tilde{X}_\Phi^{(d)}$ is the particular solution and $X_\Phi S_d(\lambda_i)$ is the homogeneous solution. This paper only deals with the calculation of the particular solutions. The calculation of the coefficient matrix $S_d(\lambda_i)$ can be found in Refs. 3, 7, and 8 and will not be discussed.

New Direct Method

The λ_i-frequency shift stiffness matrix can be transformed into a Hermite standard form[9] by the Gauss principal element elimination technique

$$P\bar{A}(\lambda_i) = H \tag{18}$$

where P is the product of the series of row operation matrix in Gauss elimination and H is a matrix of Hermite standard form.

Let

$$\bar{F}_d = PF_d(\lambda_i) \tag{19}$$

Eliminating the appropriate r columns, H becomes

$$\begin{bmatrix} I_{N-r} \\ 0 \end{bmatrix}$$

Let the corresponding rows of $\bar{F}_d$ be 0; a valid solution for $\tilde{X}_\Phi^{(d)}$ is then found.

Exchanging appropriate columns of the H from Eq. (18) can give

$$HQ = \begin{bmatrix} I_{N-r} & K \\ 0 & 0 \end{bmatrix} \tag{20}$$

where Q is the column exchange matrix. According to the theory of generalized inverses,[9] the product of Q and P is a generalized $\{1\}$ inverse of $\bar{A}(\lambda_i)$ and can be written as

$$C(\lambda_i) = QP, \qquad C(\lambda_i) \in \bar{A}(\lambda_i)\{1\} \tag{21}$$

Thus, after a particular solution of eigenvector derivative governing equation (10) is found by Gauss elimination, a generalized $\{1\}$ inverse of the λ_i-frequency shift stiffness matrix can be obtained conveniently.

Family of Modal Methods

The constraint generalized inverse of $\bar{A}(\lambda_i)$ has two expressions[7,8]:

$$G_e(\lambda_i) = \begin{cases} \left(I_N - X_\Phi Y_\Phi^T B\right) C(\lambda_i) \left(I_N - BX_\Phi Y_\Phi^T\right) & (22) \\ \displaystyle\sum_{\substack{l=1 \\ l \neq i \sim i+r-1}}^{N} \frac{x_l y_l^T}{\lambda_l - \lambda_i} & (23) \end{cases}$$

Nevertheless, the expressions of the constraint generalized inverse of $\bar{A}(\lambda_j)$ are

$$G_e(\lambda_j) = \begin{cases} \left(I_N - X_\Psi Y_\Psi^T B\right) C(\lambda_j) \left(I_N - BX_\Psi Y_\Psi^T\right) & (24) \\ \displaystyle\sum_{\substack{l=1 \\ l \neq j \sim j+n-1}}^{N} \frac{x_l y_l^T}{\lambda_l - \lambda_j} & (25) \end{cases}$$

$G_e(\lambda_i)$ can be calculated by Eq. (22) because we have already have $C(\lambda_i)$. But a fine analysis of complex structures requires a fine mesh of the finite element. In this situation, N may be a very large number. Therefore, it is very expensive to calculate $G_e(\lambda_j)$ by Eq. (24). Now we provide the technique of calculating $G_e(\lambda_j)$ from $G_e(\lambda_i)$, as the differentiated eigenvalues are always located at the low-frequency band.

Let $\rho_l(\lambda_j) = (\lambda_j - \lambda_i)/(\lambda_l - \lambda_i)$ and positive integer $m \geq \max(i+r-1, j+n-1)$. Then when $l > m$, we have $\rho_l^2(\lambda_j) < 1$. Thus, Eq. (25) can be expanded into a convergent power series of $\rho_l(\lambda_j)$:

$$\begin{aligned} G_e(\lambda_j) &= \sum_{\substack{l=1 \\ l \neq j \sim j+n-1}}^{m} \frac{x_l y_l^T}{\lambda_l - \lambda_j} + \sum_{l=m+1}^{N} \frac{x_l y_l^T}{\lambda_l - \lambda_j} \\ &= \sum_{\substack{l=1 \\ l \neq j \sim j+n-1}}^{m} \frac{x_l y_l^T}{\lambda_l - \lambda_j} + \sum_{l=m+1}^{N} \frac{x_l y_l^T}{\lambda_l - \lambda_i} \cdot \frac{1}{1 - \rho_l(\lambda_j)} \\ &= \sum_{\substack{l=1 \\ l \neq j \sim j+n-1}}^{m} \frac{x_l y_l^T}{\lambda_l - \lambda_j} + \sum_{l=m+1}^{N} \frac{x_l y_l^T}{\lambda_l - \lambda_i} \\ &\quad \times \left[1 + \rho_l(\lambda_j) + \rho_l^2(\lambda_j) + \cdots\right] \\ &= J_{-1} + J_0 + J_1 + \cdots + J_q + \cdots \end{aligned} \tag{26}$$

where

$$J_{-1} = \sum_{\substack{l=1 \\ l\neq j\sim j+n-1}}^{m} \frac{x_l y_l^T}{\lambda_l - \lambda_j} = -\frac{X_\Phi Y_\Phi^T}{\lambda_j - \lambda_i} + \sum_{\substack{l=1 \\ l\neq i\sim i+r-1 \\ l\neq j\sim j+n-1}}^{m} \frac{x_l y_l^T}{\lambda_l - \lambda_j} \tag{27a}$$

$$J_0 = \sum_{l=m+1}^{N} \frac{x_l y_l^T}{\lambda_l - \lambda_i} = G_e(\lambda_i) - \left(\frac{X_\Psi Y_\Psi^T}{\lambda_j - \lambda_i} + \sum_{\substack{l=1 \\ l\neq i\sim i+r-1 \\ l\neq j\sim j+n-1}}^{m} \frac{x_l y_l^T}{\lambda_l - \lambda_i} \right) \tag{27b}$$

$$J_1 = (\lambda_j - \lambda_i) \sum_{l=m+1}^{N} \frac{x_l y_l^T}{(\lambda_l - \lambda_i)^2} = [(\lambda_j - \lambda_i)G_e(\lambda_i)B]G_e(\lambda_i) - \left[\frac{X_\Psi T_\Psi^T}{\lambda_j - \lambda_i} + (\lambda_j - \lambda_i) \sum_{\substack{l=1 \\ l\neq i\sim i+r-1 \\ l\neq j\sim j+n-1}}^{m} \frac{x_l y_l^T}{(\lambda_l - \lambda_i)^2} \right] \tag{27c}$$

...

$$J_q = (\lambda_j - \lambda_i)^q \sum_{l=m+1}^{N} \frac{x_l y_l^T}{(\lambda_l - \lambda_i)^{q+1}} = [(\lambda_j - \lambda_i)G_e(\lambda_i)B]^q G_e(\lambda_i) - \left[\frac{X_\Psi Y_\Psi^T}{\lambda_j - \lambda_i} - (\lambda_j - \lambda_i)^q \sum_{\substack{l=1 \\ l\neq i\sim i+r-1 \\ l\neq j\sim j+n-1}}^{m} \frac{x_l y_l^T}{(\lambda_l - \lambda_j)^{q+1}} \right] \tag{27d}$$

The partial sum is introduced as

$$\begin{aligned} Dq &= \sum_{s=-1}^{q} J_s \\ &= -\frac{X_\Phi Y_\Phi^T}{\lambda_j - \lambda_i} - \frac{(q+1)X_\Psi Y_\Psi^T}{\lambda_j - \lambda_i} + \sum_{s=0}^{q} [(\lambda_j - \lambda_i)G_e(\lambda_i)B]^s G_e(\lambda_i) \\ &\quad + \sum_{\substack{l=1 \\ l\neq i\sim i+r-1 \\ l\neq j\sim j+n-1}}^{m} x_l y_l^T \left\{ \frac{1}{\lambda_l - \lambda_j} - \left[\frac{1}{(\lambda_l - \lambda_j)} + \frac{\lambda_j - \lambda_i}{(\lambda_l - \lambda_i)^2} + \frac{(\lambda_j - \lambda_i)^2}{(\lambda_l - \lambda_i)^3} + \cdots + \frac{(\lambda_j - \lambda_i)^q}{(\lambda_l - \lambda_i)^{q+1}} \right] \right\} \end{aligned} \tag{28}$$

It is easy to prove that the last sum on the right-hand side is equal to

$$\sum_{\substack{l=1 \\ l\neq i\sim i+r-1 \\ l\neq j\sim j+n-1}}^{m} \frac{x_l y_l^T}{\lambda_l - \lambda_j} \rho_l^{q+1}(\lambda_j)$$

thus we obtain

$$Dq = -\frac{X_\Phi Y_\Phi^T + (q+1)X_\Psi Y_\Psi^T}{\lambda_j - \lambda_i} + \sum_{s=0}^{q} [(\lambda_j - \lambda_i)G_e(\lambda_i)B]^s G_e(\lambda_i) + \sum_{\substack{l=1 \\ l\neq i\sim i+r-1 \\ l\neq j\sim j+n-1}}^{m} \frac{x_l y_l^T}{\lambda_l - \lambda_j} \rho_l^{q+1}(\lambda_j) \tag{29}$$

Obviously, we have

$$\lim_{q\to\infty} Dq = G_e(\lambda_j) \tag{30}$$

and the qth approximation of the constraint generalized inverse is

$$G_e(\lambda_j) \approx Dq, \qquad q = -1, 0, 1, \ldots \tag{31}$$

In particular, when $\lambda_i = 0$, Eq. (29) becomes

$$Dq = -\frac{X_R Y_R^T}{\lambda_j} - \frac{(q+1)X_\Psi Y_\Psi^T}{\lambda_j} + \sum_{s=0}^{q} [\lambda_j G_e(0)B]^s G_e(0) + \sum_{\substack{l=r+1 \\ l\neq j\sim j+n-1}}^{m} \frac{x_l y_l^T}{\lambda_l - \lambda_j} \cdot \left(\frac{\lambda_j}{\lambda_l}\right)^{q+1} \tag{32}$$

where X_R and Y_R are right and left rigid-body modes associated with zero eigenvalues. This is the formula derived in Ref. 7 for calculating $G_e(\lambda_j)$ from $G_e(0)$.

For the eigenproblem

$$AX_\Psi = BX_\Psi \Lambda_\Psi \tag{33}$$

the d-order eigenvector governing equation is

$$\bar{A}(\lambda_j)X_\Psi^{(d)} = F_d(\lambda_j), \qquad d = 1, 2, \ldots \tag{34}$$

Corresponding to Eqs. (11–15) we have

$$\bar{A}_d(\lambda_j) = A^{(d)} - \lambda_j B^{(d)} \tag{35}$$

$$F_d(\lambda_j) = \sum_{e=1}^{d} \frac{d!}{e!(d-e)!} \times \left[\sum_{h=1}^{e} \frac{e!}{h!(e-h)!} B^{(e-h)} X_\Psi \Lambda_\Psi^{(h)} - \bar{A}_e(\lambda_j) X_\Psi^{(d-e)} \right] \tag{36}$$

$$\Lambda_\Psi^{(d)} = -Y_\Psi \Theta_d(\lambda_j) \tag{37}$$

$$\Theta_d(\lambda_j) = F_d(\lambda_j) - BX_\Psi \Lambda_\Psi^{(d)} \tag{38}$$

The general solution of Eq. (33) is

$$X_\Psi^{(d)} = \tilde{X}_\Psi^{(d)} + X_\Psi S_d(\lambda_j) \tag{39}$$

and the particular solution is[7,8]

$$\tilde{X}_\Psi^{(d)} = G_e(\lambda_j)\Theta_d(\lambda_j) \approx Dq\Theta_d(\lambda_j) \tag{40}$$

This particular solution is orthogonal to X_Ψ with respect to B.

Especially, when $d = 1$, we have

$$\tilde{X}_\Psi^{(1)} \approx Dq[-A_1(\lambda_j)X_\psi] \tag{41}$$

In Eq. (29), most of computation is for

$$\sum_{s=0}^{q} [(\lambda_j - \lambda_i)G_e(\lambda_i)B]^s G_e(\lambda_i)$$

whereas a small integer for q (such as 1 or 2) can give a rather accurate solution, the total computational cost of Eq. (29) is less than that of Eq. (24), when the DOF is large and several eigenvector derivatives with different sets of repeated eigenvalues are wanted.

Example

The planar grillage of Fig. 1 is composed of eight beam elements. The total DOF is 15; there are five knots, and each has 3 DOF in the plan. Table 1 shows the design data; A_1–A_8 are the section areas of the elements.

Figure 2 is a plot of the four lowest eigenvalues of the structure as a function of the design variable p, as p varies from 0.6 to 1.4 cm. Note that λ_2, λ_3, and λ_4 cross at $p = 1$ cm.

A second-order approximation of the four lowest eigenvalues is formed using the sensitivities. These second-order approximations are plotted in Fig. 2 along with the independently calculated eigenvalues with SAP. In calculating the second-order eigenvalue derivatives, the first-order eigenvector derivatives are needed.[3,7,8] Figure 2 shows that the calculated sensitivities are very accurate.

Table 1 Example problem parameters

Young's modulus	3.0×10^7 N/cm^2
Density	0.074 kg/cm^3
L	72 cm
I_1, I_2, I_3, I_4	$p^4/12$
I_5, I_6, I_7, I_8	1/12 cm^4
A_1, A_2	p^2
$A_3, A_4, A_5, A_6, A_7, A_8$	1 cm^2
p_0	1 cm

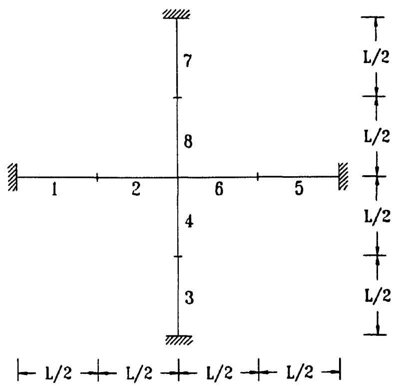

Fig. 1 Planar grillage.

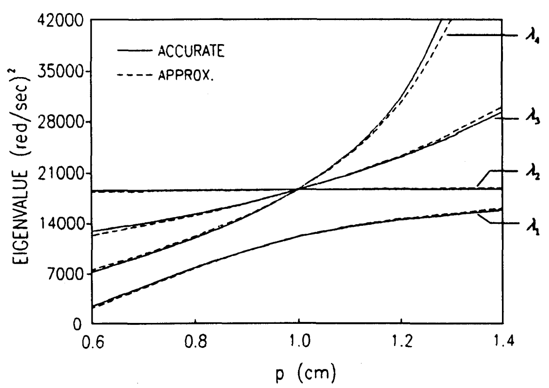

Fig. 2 Four lowest eigenvalues with approximations.

We get $\tilde{x}_1^{(1)}$ by the direct method, and then $C(\lambda_1)$, a generalized {1} inverse of $\bar{A}(\lambda_1)$, is found with Eq. (21) and $G_e(\lambda_1)$, the constraint generalized inverse of $\bar{A}(\lambda_1)$, is found with Eq. (22). In Eq. (29), let $m = 9$ and $q = 2$, the approximation of $G_e(\lambda_2)$ is computed from $G_e(\lambda_1)$. Then $\tilde{X}_\Psi^{(1)}$ can be obtained with Eq. (41). A first-order approximation of the four eigenvectors is computed with Eq. (39).

Conclusion

A combined method has been developed for calculation of eigenvector derivatives in the case of repeated eigenvalues, and the technique of calculating the approximate constraint generalized inverse has been presented. A simple example problem supports the applicability of the technique.

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References

[1]Ting, T., "Accelerated Subspace Iteration for Eigenvector Derivatives," *AIAA Journal*, Vol. 30, No. 8, 1992, pp. 2114–2118.

[2]Nelson, R. B., "Simplified Calculations of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, No. 9, 1976, pp. 1201–1205.

[3]Mills-Curran, W. C., "Calculation of Eigenvector Derivatives for Structures with Repeated Eigenvalues," *AIAA Journal*, Vol. 26, No. 7, 1988, pp. 867–871.

[4]Fox, R. L., and Kapoor, M. D., "Rates of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426–2429.

[5]Wang, B. P., "An Improved Approximate Method for Computing Eigenvector Derivatives in Structural Dynamics," *AIAA Journal*, Vol. 29, No. 6, 1991, pp. 1018–1020.

[6]Akgün, M. A., "New Family of Modal Method for Calculating Eigenvector Derivatives," *AIAA Journal*, Vol. 32, No. 2, 1994, pp. 379–386.

[7]Wang, W. L., "A Family of Modal Methods for Computing Eigenvector Derivatives with Repeated Roots," *Acta Mechanica Sinica*, Vol. 12, No. 2, 1996, pp. 154–168.

[8]Zhang, Y. Q., and Wang, W. L., "Eigenvector Derivatives of Generalized Nondefective Eigenvector with Repeated Eigenvalues," *Journal of Engineering for Gas Turbines and Power*, Vol. 117, No. 1, 1995, pp. 207–212.

[9]Ben-Israel, A., and Greville, T. N. E., *Generalized Inverses, Theory and Applications*, Wiley, New York, 1974.

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